CHEVALLEY BASES FOR LIE MODULES (1)

BY
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1. **Introduction.** In Theorem 1 of [3], Chevalley established the existence of a basis for a complex semi-simple Lie algebra $\mathfrak{L}_C$ with certain special properties. One property is of integral multiplication table, making it possible to define an analogous Lie algebra $\mathfrak{L}$ over an arbitrary field K. Another (related) property is the fact that the linear transformations $(n!)^{-1}(\operatorname{ad} e_\alpha)^n$ (where e_α is a root vector in the basis and n a positive integer) have integral matrices, making it possible to associate the automorphisms of $\mathfrak{L}$ with those of $\mathfrak{L}_C$ (see e.g. [11], [12]). We will observe first that this latter property characterizes Chevalley bases (in an appropriate sense). Thought of as a property of the adjoint representation, it has a natural generalization to other representations, and will be used to define Chevalley bases for an $\mathfrak{L}_C$-module. §3 and 4 will be devoted to proving the existence of such a basis for a finite-dimensional irreducible $\mathfrak{L}_C$-module $\mathfrak{M}_C$.

Since the original preparation of this paper, R. Ree has published a paper [13] which contains a proof of the existence of Chevalley bases for modules. His proof depends on the Cartan classification of the algebras $\mathfrak{L}_C$, and he states, "a direct proof of [the existence] is desirable." While Theorem 2 of the present paper is not quite as strong as Ree's Theorem (1.6), the proof presented here is direct and entirely different from Ree's. Furthermore, essentially the same consequences for algebras, modules, and groups over arbitrary fields can be obtained *except* that Theorem 2 below is useless in passing to a field of characteristic 2 or 3 for some purposes (see §5). It should also be noted that Ree's proof depends on Chevalley's basis theorem, and the present proof does not.

A Chevalley basis for $\mathfrak{M}_C$ will be used to associate with $\mathfrak{M}_C$ (or with the corresponding representation R) an $\mathfrak{L}$-module $\mathfrak{M}$ and a linear group G^R analogous to the Chevalley group G' of automorphisms of $\mathfrak{L}$. A general fixed point theorem for the groups G^R, analogous to Theorem 1 of [11], is proved in §6. Application of this theorem to the case of a simple Lie algebra of type F_4 shows that every automorphism of an exceptional central simple Jordan algebra over an arbitrary field of characteristic $\neq 2, 3$ has at least a three-dimensional fixed point space. Other applications of the theorem yield known results about fixed points of rotations in odd-dimensional spaces and fixed points of automorphisms of Cayley algebras.

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2. **A characterization of Chevalley bases.** Let $\mathfrak{L}$ be a finite-dimensional complex semi-simple Lie algebra, and let $\mathfrak{H}$ be a Cartan subalgebra. Let $\{e_\alpha\}$ be a complete set of root vectors for the (nonzero) roots α of $\mathfrak{L}$ with respect to $\mathfrak{H}$. We write

$$[e_\alpha e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} \tag{1}$$

if $\alpha + \beta$ is a root. Set $N_{\alpha,\beta} = 0$ if $\alpha + \beta$ is not a root (and not zero), and then (1) holds for all pairs α, β if $\alpha \neq -\beta$. We will denote the α-string of roots through β by

$$\beta - r_{\alpha\beta}\alpha, \cdots, \beta, \beta + \alpha, \cdots, \beta + q_{\alpha\beta}\alpha.$$

Let $\mathfrak{L}_\alpha$ denote the root space of α, and let h_α be the unique element of $[\mathfrak{L}_\alpha, \mathfrak{L}_{-\alpha}]$ such that $\alpha(h_\alpha) = 2$.

A set $\{e_\alpha\}$ of root vectors can always be replaced by a set of scalar multiples of themselves so that the new set satisfies

$$[e_\alpha e_{-\alpha}] = h_\alpha. \tag{2}$$

Assuming this done, we have [3, pp. 21-23]

$$N_{\alpha,\beta} N_{-\alpha,-\beta} = (r_{\alpha\beta} + 1)^2. \tag{3}$$

(The reason for a difference in sign between (3) and the formula given in [3] is that Chevalley takes the adjoint mappings to act on the left instead of on the right.)

The essence of the Chevalley basis theorem [3, p. 24] is that the e_α may again be replaced by scalar multiples of themselves, preserving (2) and so that $N_{\alpha,\beta} = N_{-\alpha,-\beta}$. Hence $[e_\alpha e_\beta] = \pm (r_{\alpha\beta} + 1)\, e_{\alpha+\beta}$. The root vectors can then be taken as part of a basis having an integral multiplication table, and having the further property that the matrices of the linear transformations $(n!)^{-1}(\operatorname{ad} e_\alpha)^n$, for all roots α and positive integers n, have only integer entries. This property, in the presence of (2), characterizes Chevalley bases, and in fact can be stated in a somewhat weaker form. Henceforth let Π denote a fixed fundamental system of roots.

THEOREM 1. *Let $\mathfrak{L}$ have a basis consisting of a basis for $\mathfrak{H}$ and a complete set $\{e_\beta\}$ of root vectors such that*

(a) $[e_\beta e_{-\beta}] = h_\beta$ *for all roots* β;

(b) *the matrices of the linear transformations* $(n!)^{-1}(\operatorname{ad} e_\alpha)^n$ *have integer entries for* $\pm \alpha \in \Pi$, $n = 1, 2, \cdots$.

Then $N_{\alpha,\beta} = N_{-\alpha,-\beta}$ for all pairs of roots α, β, and hence the e_β's are root vectors in a Chevalley basis.

Proof. Let $\alpha \in \Pi$, and let β be a root such that $\alpha + \beta$ is not a root, i.e., $q_{\alpha\beta} = 0$ and $r_{\alpha\beta}$ is some integer r. Then for $k = 1, 2, \cdots, r$, we have

(4) $$e_\beta(\operatorname{ad} e_{-\alpha})^k = N_{\beta,-\alpha}N'_{\beta-\alpha,-\alpha}\cdots N_{\beta-(k-1)\alpha,-\alpha}e_{\beta-k\alpha},$$

(5) $$e_{-\beta}(\operatorname{ad} e_\alpha)^k = N_{-\beta,\alpha}N_{-\beta+\alpha,\alpha}\cdots N_{-\beta+(k-1)\alpha,\alpha}e_{-\beta+k\alpha}.$$

The N's are all integers ((b) for $n = 1$) and (a) implies that (3) holds. We prove by induction that

(6) $$N_{-\beta+k\alpha,\alpha} = N_{\beta-k\alpha,-\alpha} = \pm (k+1), \qquad k = 0, 1, \cdots, r-1.$$

For $k = 0$ we have $N_{-\beta,\alpha}N_{\beta,-\alpha} = (r_{-\alpha\beta}+1)^2 = 1$, and hence $N_{-\beta,\alpha} = N_{\beta,-\alpha} = \pm 1$. Assuming (6) for $k-1$, (4) and (5) give

(7) $$e_\beta(\operatorname{ad} e_{-\alpha})^{k+1} = \pm k!\, N_{\beta-k\alpha,-\alpha}e_{\beta-(k+1)\alpha},$$

(8) $$e_{-\beta}(\operatorname{ad} e_\alpha)^{k+1} = \pm k!\, N_{-\beta+k\alpha,\alpha}e_{-\beta+(k+1)\alpha}.$$

Thus (b) implies that $k+1$ divides both $N_{\beta-k\alpha,-\alpha}$ and $N_{-\beta+k\alpha,\alpha}$. Since

$$N_{\beta-k\alpha,-\alpha}N_{-\beta+k\alpha,\alpha} = (r_{-\alpha,\beta-k\alpha}+1)^2 = (k+1)^2,$$

(6) follows for k, which completes the induction step.

Now consider an arbitrary root β with α-string

$$\beta - r\alpha, \cdots, \beta, \cdots, \beta + q\alpha.$$

Let $\beta' = -\beta + r\alpha$. Then $\beta' + \alpha$ is not a root, and

$$N_{\beta,\alpha} = N_{-\beta'+r\alpha,\alpha} = \pm (r+1) = N_{\beta'-r\alpha,-\alpha} = N_{-\beta,-\alpha}.$$

Thus we have $N_{\beta,\alpha} = N_{-\beta,-\alpha}$ for β an arbitrary root and $\alpha \in \Pi$. Since β can be replaced by $-\beta$, this also holds for $-\alpha \in \Pi$. Consider the roots ordered lexicographically by Π. Our result thus far is the initial step for proving the desired conclusion for an arbitrary $\alpha > 0$ (and hence for all α) by induction on the ordering.

Thus let $\gamma > 0$ be arbitrary, but not in Π, and assume $N_{\beta,\alpha} = N_{-\beta,-\alpha}$ for all β and all $\alpha < \gamma$. Write $\gamma = \alpha + \delta$, where α and δ are positive roots $< \gamma$. For an arbitrary root β, the Jacobi identity for $e_\beta, e_\alpha, e_\delta$ yields

(9) $$N_{\beta,\gamma}N_{\delta,\alpha} = N_{\beta,\delta}N_{\beta+\delta,\alpha} + N_{\alpha,\beta}N_{\alpha+\beta,\delta}.$$

Similarly, for $e_{-\beta}$, $e_{-\alpha}, e_{-\delta}$, we get

(10) $$N_{-\beta,-\gamma}N_{-\delta,-\alpha} = N_{-\beta,-\delta}N_{-\beta-\delta,-\alpha} + N_{-\alpha,-\beta}N_{-\alpha-\beta,-\delta}.$$

In all the factors of (10) except the first we have a subscript of $-\alpha$ or $-\delta$, and the induction hypothesis gives

(11) $$N_{-\beta,-\gamma}N_{\delta,\alpha} = N_{\beta,\delta}N_{\beta+\delta,\alpha} + N_{\alpha,\beta}N_{\alpha+\beta,\delta}.$$

Since $\alpha + \delta$ is a root, namely γ, $N_{\delta,\alpha} \neq 0$, and (9) and (11) imply $N_{\beta,\gamma} = N_{-\beta,-\gamma}$, which completes the induction step, and the theorem.

3. **Module bases.** In this section $\mathfrak{L}$ will denote a finite-dimensional split

semi-simple Lie algebra over an arbitrary field K of characteristic 0. Let (A_{ij}) be the Cartan matrix of $\mathfrak{L}$ and e_i, f_i, h_i $(1 \leqq i \leqq l)$ a set of canonical generators [8, p. 126] for $\mathfrak{L}$. Let $\mathfrak{M}$ be a finite-dimensional irreducible $\mathfrak{L}$-module with associated representation R.

Definition. A *Chevalley basis* for $\mathfrak{M}$ is any basis of weight vectors with respect to which the linear transformations

$$(12) \qquad (m!)^{-1}(e_i^R)^m, (m!)^{-1}(f_i^R)^m, \qquad 1 \leqq i \leqq l, \quad m = 1, 2, \cdots,$$

have matrices with integer entries.

Remark. In [13], Ree calls a basis for $\mathfrak{M}$ *regular* if it satisfies the condition above for all the root vectors of a Chevalley basis of $\mathfrak{L}$, not just the canonical generators. It is for this reason that [13, (1.6)] is stronger than the following theorem.

Theorem 2. *Every finite-dimensional irreducible $\mathfrak{L}$-module $\mathfrak{M}$ has a Chevalley basis.*

The purpose of this section and the next will be the proof of this theorem. The main ideas in the proof will be sketched first, and the computational details, isolated as Lemma 1 below, will be deferred until §4.

We begin by constructing $\mathfrak{M}$ as in Jacobson [8, Chapter VII]. Let $\mathfrak{F}$ be the free Lie algebra on $3l$ generators e_i, f_i, h_i $(1 \leqq i \leqq l)$, and let $\tilde{\mathfrak{L}} = \mathfrak{F}/\mathfrak{I}$, where $\mathfrak{I}$ is the ideal generated by the elements

$$(13) \qquad \begin{aligned} &[h_i h_j], \\ &[e_i f_j] - \delta_{ij} h_i, \\ &[e_i h_j] - A_{ji} e_i, \\ &[f_i h_j] + A_{ji} f_i. \end{aligned}$$

Let $\mathfrak{H}$ be the span of the h_i's in $\mathfrak{F}$, and let α_i be the linear function on $\mathfrak{H}$ such that $\alpha_i(h_j) = A_{ji}$, $i, j = 1, 2, \cdots, l$. Let Λ be a dominant integral linear function on $\mathfrak{H}$, i.e., $\Lambda(h_i)$ is a non-negative integer for $i = 1, 2, \cdots, l$. Let $\mathfrak{X}$ be the free algebra (associative with identity) on l generators $x_1, x_2, \cdots, x_l$. $\mathfrak{X}$ becomes a module for $\mathfrak{F}$ and for $\tilde{\mathfrak{L}}$ by the definitions

$$(14) \qquad (x_{i_1} \cdots x_{i_r}) h_i = (\Lambda - \alpha_{i_1} - \cdots - \alpha_{i_r})(h_i) x_{i_1} \cdots x_{i_r},$$

$$(15) \qquad (x_{i_1} \cdots x_{i_r}) f_i = x_{i_1} \cdots x_{i_r} x_i,$$

$$1 e_i = 0,$$

$$(16) \qquad (x_{i_1} \cdots x_{i_r}) e_i = ((x_{i_1} \cdots x_{i_{r-1}}) e_i) x_{i_r} - \delta_{i, i_r} (\Lambda - \alpha_{i_1} - \cdots - \alpha_{i_{r-1}})(h_i) x_{i_1} \cdots x_{i_{r-1}}.$$

Here we understand that $x_{i_1} \cdots x_{i_r} = 1$ if $r = 0$. The algebra $\mathfrak{L} = \tilde{\mathfrak{L}}/\mathfrak{I}'$, where $\mathfrak{J}'$ is the intersection of the kernels of all finite-dimensional irreducible representations of $\tilde{\mathfrak{L}}$. The same names for the generators e_i, f_i, h_i have been used throughout because the space spanned by them is mapped isomorphically in passing from $\mathfrak{F}$ to $\mathfrak{L}$. $\mathfrak{X}$ has a unique maximal submodule $\mathfrak{P}$; $\mathfrak{X}/\mathfrak{P}$ is irreducible and finite-dimensional, hence is an $\mathfrak{L}$-module $\mathfrak{M}$. This turns out to be the unique finite-dimensional irreducible $\mathfrak{L}$-module with highest weight Λ.

A basis for $\mathfrak{M}$ may be found by selecting a linearly independent set of cosets modulo $\mathfrak{P}$ of monomials $x_{i_1} \cdots x_{i_r}$ in $\mathfrak{X}$. Modification of such a basis by suitable scalar multiplications will yield a Chevalley basis. Specifically, write an arbitrary monomial as $x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}$, where the highest possible exponents are displayed. We call $(k_1! \cdots k_r!)^{-1} x_{i_1}^{k_1} \cdots x_{i_r}^{k_r} a$ "modified monomial."

LEMMA 1. *The image of a modified monomial under each of the mappings* (12) *is an integral linear combination of modified monomials.*

Now specialize the base field K to the rational field Q. Modified monomials in $\mathfrak{X}$ are weight vectors (by (14)). For each weight λ, let $\mathfrak{Y}_\lambda$ be the additive group generated by the cosets of the modified monomials belonging to λ. $\mathfrak{Y}_\lambda$ is finitely generated, and hence has a basis. Such a basis is also a basis for the Q-space spanned by $\mathfrak{Y}_\lambda$, which is the weight space $\mathfrak{M}_\lambda$ of $\mathfrak{M}$. We take as basis for $\mathfrak{M}$ the union of these bases for weight spaces. Since every basis element for $\mathfrak{M}$ is an integral linear combination of cosets of modified monomials, and every such coset is an integral linear combination of basis elements, the mappings (12) have integral matrices with respect to the basis chosen for $\mathfrak{M}$.

The result is extended to an arbitrary base field K of characteristic 0 by identifying the prime field of K with Q and constructing $\mathfrak{L}_K$ and $\mathfrak{M}_K$. $\mathfrak{L}_K$ is the split semi-simple Lie algebra over K with Cartan matrix (A_{ij}), and since $\mathfrak{M}$ is absolutely irreducible [8, p. 223], $\mathfrak{M}_K$ is the finite-dimensional irreducible $\mathfrak{L}_K$-module with highest weight Λ. The Chevalley basis for $\mathfrak{M}$ is also one for $\mathfrak{M}_K$. The proof of Lemma 1 is all that remains to complete the proof of Theorem 2.

REMARK. The combination of Theorems 1 and 2 does not yield an alternative proof of the Chevalley basis theorem, because of the necessity of hypothesis (a) in Theorem 1, which does not seem to have a module analogue. The combination does provide a somewhat weaker theorem which is adequate for defining the groups of Chevalley—but only on the basis of Chevalley's own results on the generation of these groups [3, Lemmas III.4 and IV.3].

4. **Proof of Lemma 1.** Half of the lemma is quite easy to prove. Let

$$(k_1! \cdots k_r!)^{-1} x_{i_1}^{k_1} \cdots x_{i_r}^{k_r}$$

be a modified monomial. We have

$$(k_1! \cdots k_r!)^{-1} x_{i_1}^{k_1} \cdots x_{i_r}^{k_r} (m!)^{-1} (f_i^R)^m = (k_1! \cdots k_r! m!)^{-1} x_{i_1}^{k_1} \cdots x_{i_r}^{k_r} x_i^m,$$

which gives a coefficient of 1 if $i \neq i_r$; otherwise, we have

$$(k_1! \cdots k_r! m!)^{-1} x_{i_1}^{k_1} \cdots x_{i_{r-1}}^{k_{r-1}} x_{i_r}^{k_r+m}$$

$$= \binom{k_r + m}{k_r} (k_1! \cdots k_{r-1}! \, (k_r + m)!)^{-1} x_{i_1}^{k_1} \cdots x_{i_{r-1}}^{k_{r-1}} x_{i_r}^{k_r+m}.$$

Thus the statement is correct for the f_i's.

In the sequel we will have to frequently write linear combinations

$$k_1 \alpha_{i_1} + \cdots + k_r \alpha_{i_r}$$

of roots, sometimes with complicated subscripts. We denote such a linear combination by $k\alpha_i[1,r]$. If each $k_i = 1$, we write $\alpha_i[1,r]$.

It is useful to replace (16) with the explicit form for the operation of e_i on a monomial, which is:

$$(17) \qquad (x_{i_1} \cdots x_{i_r}) e_i = - \sum_{j=1}^{r} \delta_{i_j i} (\Lambda - \alpha_i[1, j-1])(h_i) x_{i_1} \cdots \hat{x}_{i_j} \cdots x_{i_r},$$

where $\hat{}$ denotes deletion of the argument. This follows immediately from (16) by induction on r. Thus the image of a monomial under e_i is a linear combination of monomials each of which is obtained from the original by deletion of exactly one x_i.

Next we note the significance of having more than one x_i appearing together in a monomial. On a monomial of the form $x_{i_1} \cdots x_i^k \cdots x_{i_r}$, operation by e_i produces (inter alia) k terms all involving the monomial

$$x_{i_1} \cdots x_i^{k-1} \cdots x_{i_r},$$

which can thus be combined. If x_{i_j} is the first x_i in the group being considered, the sum of the coefficients of these terms is:

$$(18) \qquad \begin{aligned} &- k(\Lambda - \alpha_i[1, j-1])(h_i) - \sum_{m=0}^{k-1} m\alpha_i(h_i) \\ &\qquad = - k\{(\Lambda - \alpha_i[1, j-1])(h_i) - k + 1\}. \end{aligned}$$

The next task is to compute explicitly the effect of $(e_i^R)^m$ on an arbitrary monomial. To do this, we need to display all the powers of x_j's appearing in the monomial, singling out the x_i's. The reader is asked to visualize a monomial X with sequence of subscripts

$$(19) \qquad i_1, \cdots, i_{r_1}; i; i_{r_1+1}, \cdots, i_{r_2}; i; i_{r_2+1}, \cdots, i_{r_n}; i; i_{r_n+1}, \cdots, i_{r_{n+1}},$$

and corresponding sequence of exponents

(20) $k_1, \cdots, k_{r_1}; N_1; k_{r_1+1}, \cdots, k_{r_2}; N_2; k_{r_2+1}, \cdots, k_{r_n}; N_n; k_{r_n+1}, \cdots, k_{r_{n+1}}$,

where we may have $r_1 = 0$ and/or $r_{n+1} = r_n$, but adjacent subscripts must be distinct. Now Xe_i^m will be a linear combination of monomials, a typical one of which can be obtained from X by replacing $x_i^{N_j}$ by $x_i^{N_j - m_j}$ $(1 \leqq j \leqq n)$, where $m_1 + m_2 + \cdots + m_n = m$. In other words, a typical term of Xe_i^m is associated with an "ordered partition" $m = m_1 + \cdots + m_n$ of the integer m, subject to $0 \leqq m_j \leqq N_j$ for all j, and in fact there is a term of Xe_i^m for each such ordered partition (possibly with zero coefficient, or course).

We need to consider the α_i-strings of weights through each of the weights $\lambda_j = \Lambda - k\alpha_i[1, r_j] - (\sum_1^j N_v)\alpha_i$, $1 \leqq j \leqq n$. The jth such string will start with $\lambda_j - p_j\alpha_i$ and end with $\lambda_j + q_j\alpha_i$. Then we have [8, Theorem 4.1]:

$$(\Lambda - k\alpha_i[1, r_j])(h_i) = p_j - q_j + 2\sum_1^j N_v. \tag{21}$$

For convenience, we introduce the following abbreviations for expressions which recur frequently:

$$\begin{aligned} T_j &= p_j - q_j + N_j, \\ M_j &= m_1 + m_2 + \cdots + m_j \qquad (M_0 = 0, M_n = m), \\ A_j &= T_j + M_j, \\ B_j &= T_j + M_{j-1}, \\ C_j &= A_j + M_{j-1} = B_j + M_j, \end{aligned}$$

where in each case $j = 1, 2, \cdots, n$.

We prove by induction on m that the term of Xe_i^m corresponding to the ordered partition $m = m_1 + \cdots + m_n$ has the coefficient

$$(-1)^m m! \prod_{j=1}^n \left[\prod_{t=0}^{m_j - 1} (N_j - t) \binom{A_j}{B_j} \right]. \tag{22}$$

For $m = 1$, we have some $m_j = 1$, all others $= 0$. The assertion is that we get a coefficient of $-N_j(T_j + 1)$. By (18), we get a coefficient of

$$\begin{aligned} &- N_j \left\{ \left(\Lambda - k\alpha_i[1, r_j] - \sum_1^{j-1} N_v \alpha_i \right) (h_i) - N_j + 1 \right\} \\ &= - N_j \left(p_j - q_j + 2\sum_1^j N_v - 2\sum_1^{j-1} N_v - N_j + 1 \right) \qquad \text{(by (21))} \\ &= - N_j(p_j - q_j + N_j + 1) = - N_j(T_j + 1), \end{aligned}$$

as required.

Now assume the assertion is true for $m-1$. There are exactly n terms of Xe_i^{m-1} which contribute to the term of Xe_i^m under consideration, namely, those corresponding to the ordered partitions $m-1=m_1+\cdots+m_{s-1}+(m_s-1)+m_{s+1}+\cdots+m_n$. By the induction hypothesis, the sth one of these has coefficient

$$(23)\qquad \begin{gathered}(-1)^{m-1}(m-1)!\prod_{t=0}^{m_1-1}(N_1-t)\cdots\prod_{t=0}^{m_s-2}(N_s-t)\cdots\prod_{t=0}^{m_n-1}(N_n-t)\\ \cdot\binom{A_1}{B_1}\cdots\binom{A_{s-1}}{B_{s-1}}\binom{A_s-1}{B_s}\binom{A_{s+1}-1}{B_{s+1}-1}\cdots\binom{A_n-1}{B_n-1}.\end{gathered}$$

The contribution of the term with coefficient (23) to Xe_i^m is, by (18) and (21), another factor of:

$$(24)\qquad \begin{aligned}-(N_s-m_s+1)&\left\{\left(\Lambda-k\alpha_i[1,r_s]-\sum_1^{s-1}(N_v-m_v)\alpha_i\right)(h_i)-N_s+m_s\right\}\\ &=-(N_s-m_s+1)\left(p_s-q_s+2\sum_1^{s}N_v-2\sum_1^{s-1}(N_v-m_v)-N_s+m_s\right)\\ &=-(N_s-m_s+1)(T_s+2M_{s-1}+m_s)\\ &=-(N_s-m_s+1)C_s.\end{aligned}$$

The coefficient we seek is obtained by summing the product of (23) and (24) for $s=1,2,\cdots,n$. Clearly the following factors are common to every term of the sum:

$$(25)\qquad (-1)^m(m-1)!\prod_{j=1}^{n}\prod_{t=0}^{m_j-1}(N_j-t).$$

The rest of the sum, which must be factored into the remaining factors of (22), is:

$$(26)\qquad \begin{aligned}\sum_{s=1}^{n}C_s&\binom{A_1}{B_1}\cdots\binom{A_{s-1}}{B_{s-1}}\binom{A_s-1}{B_s}\binom{A_{s+1}-1}{B_{s+1}-1}\cdots\binom{A_n-1}{B_n-1}\\ &=\sum_{s=1}^{n}C_s\prod_1^{s-1}A_w!\prod_s^n(A_w-1)!\left\{\prod_1^s B_w!\prod_{s+1}^n(B_w-1)!\,(m_s-1)!\prod_{w\neq s}m_w!\right\}^{-1}\\ &=\prod_1^n(B_w!\,m_w!)^{-1}\sum_{s=1}^n m_s\left(\prod_{s+1}^n B_w\right)C_s\prod_1^{s-1}A_w!\prod_s^n(A_w-1)!\\ &=\prod_1^n(A_w-1)!\,(B_w!\,m_w!)^{-1}\sum_{s=1}^n m_sA_1\cdots A_{s-1}C_sB_{s+1}\cdots B_n.\end{aligned}$$

We now proceed to factor the sum remaining in (26). We note first that $C_1=T_1+m_1=T_1+M_1=A_1$, and one can verify from the definitions that

$$(27) \qquad m_w C_w = M_w A_w - M_{w-1} B_w, \qquad w = 1, 2, \cdots, n.$$

Then the sum in (26) becomes

$$\sum_{s=1}^{n} M_s A_1 A_2 \cdots A_s B_{s+1} \cdots B_n - \sum_{w=0}^{n-1} M_w A_1 \cdots A_w B_{w+1} \cdots B_n$$
$$= M_n A_1 A_2 \cdots A_n = m A_1 A_2 \cdots A_n.$$

Hence (26) becomes

$$(28) \qquad \left(m \prod_1^n A_w! \right) \Big/ \left(\prod_1^n B_w! \, m_w! \right) = m \prod_1^n \binom{A_w}{B_w}.$$

The product of (28) and (25) gives (22) as required to complete the induction step.

The modified monomial corresponding to the monomial X defined by (19) and (20) is $(k_1! \cdots k_{r_{n+1}}! N_1! \cdots N_n!)^{-1} X$. The monomial in the term of Xe_i^m corresponding to $m = m_1 + \cdots + m_n$ gets replaced by

$$(k_1! \ldots k_{r_{n+1}}! (N_1 - m_1)! \ldots (N_n - m_n)!)^{-1}$$

times itself. Thus, the coefficient of this term must be an integral multiple of $m! \prod_{j=1}^{n} \prod_{t=0}^{m_j - 1} (N_j - t)$, which is precisely what (22) asserts it is.

This completes the proof of Lemma 1 and hence also of Theorem 2.

5. **Analogues of the Chevalley groups.** In this section the notation of [11, §2] will be used. In particular, $\mathfrak{L}_C$ denotes a complex semi-simple Lie algebra with a fixed Chevalley basis $\{h_i, e_\alpha\}$, and $\mathfrak{L}$ the corresponding Lie algebra of Chevalley over an arbitrary field K of characteristic $\neq 2, 3$. (This restriction on characteristics will be assumed henceforth without further mention.)

If α, β and $\alpha + \beta$ are roots, then

$$[e_\alpha e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta},$$

where $N_{\alpha,\beta}$ is an integer, and $1 \leqq |N_{\alpha,\beta}| \leqq 4$. Thus each e_α can be written as a rational multiple of a product of the form $[\cdots[[e_{i_1} e_{i_2}] e_{i_3}] \cdots e_{i_r}]$ or of the form $[\cdots[[f_{i_1} f_{i_2}] f_{i_3}] \cdots f_{i_r}]$, where the multiplier (in lowest terms) has only powers of 2 and 3 in the denominator. It will be convenient in the sequel to refer to the subring B of the rational field Q consisting of all rationals having only 2 and 3 as prime factors of their denominators (when in the lowest terms). If p is a prime > 3, then B/pB is a field of p elements and will be identified with the prime field K_0 of any field K of characteristic p.

Let Λ be a dominant integral linear function on $\mathfrak{H}_C$ with respect to the basis $h_1, h_2, \cdots, h_l$. Let $\mathfrak{M}_C$ be the corresponding finite-dimensional irreducible $\mathfrak{L}_C$-module, and R the associated representation. Let $y_1, \cdots, y_n$ be a Chevalley basis for $\mathfrak{M}_C$, in the sense defined above, which will henceforth be fixed. When convenient, linear transformations in $\mathfrak{M}_C$ will be identified with matrices relative to this basis.

We now describe a process for associating with the $\mathfrak{L}_C$-module $\mathfrak{M}_C$ an $\mathfrak{L}$-module $\mathfrak{M}$. The matrices e_i^R, f_i^R, h_i^R have integer entries, and by the remarks above, e_α^R is a matrix with entries in B for every root vector e_α in the Chevalley basis. Reduce the entries of each of the matrices h_i^R, e_α^R modulo the characteristic of K. The result is a set of matrices over the prime field K_0 of K. Considering the elements $\{h_i, e_\alpha\}$ as the Chevalley basis of $\mathfrak{L}$, and continuing to denote the matrices over K_0 by h_i^R, e_α^R, we have a linear mapping R: $h_i \rightarrow h_i^R$, $e_\alpha \rightarrow e_\alpha^R$ of $\mathfrak{L}$ into the space of n-by-n matrices over K.

It is easy to verify that R is a matrix representation of $\mathfrak{L}$, and as such has an associated module $\mathfrak{M}$, considering the matrices in $\mathfrak{L}^R$ as linear transformations of the K-space $\mathfrak{M}$ with respect to a fixed basis. We have $\dim_K \mathfrak{M} = n = \dim_C \mathfrak{M}_C$, and it is convenient to identify the Chevalley basis $y_1, \cdots, y_n$ of $\mathfrak{M}_C$ with the fixed basis in $\mathfrak{M}$.

We recall the definition of the Chevalley group G'. Let

$$x(t;\ \alpha) = \exp(t \operatorname{ad} e_\alpha),$$

for $t \in C$ and α a root of $\mathfrak{L}_C$. (The notation $x_\alpha(t)$ is used in [11].) $x(t;\ \alpha)$ is an automorphism of $\mathfrak{L}_C$ and has a matrix (relative to the Chevalley basis) with entries which are polynomials in t with integer coefficients. Replace the complex parameter t by an indeterminate, and then specialize the indeterminate to an arbitrary element $t \in K$. The result, also denoted $x(t;\ \alpha)$, is the matrix relative to the Chevalley basis of an automorphism of $\mathfrak{L}$. The group G' is the group generated by the automorphisms $x(t;\ \alpha)$ for all roots α of $\mathfrak{L}_C$ and all $t \in K$. Actually, G' is generated by the $x(t;\ \alpha)$ for $t \in K$ and $\pm\alpha \in \Pi$, a fixed fundamental system of roots [3, p. 48].

We now define groups similar to G' by the use of Theorem 2. Let $x(t;\ \alpha, R) = \exp te_\alpha^R$, $\pm\alpha \in \Pi$, $t \in C$, where $\Pi = \{\alpha_1, \alpha_2, \cdots, \alpha_l\}$. Relative to the Chevalley basis of $\mathfrak{M}_C$, each $x(t;\ \alpha, R)$ has a matrix of determinant 1 with entries which are polynomials in t with integer coefficients. As above, replace t by an indeterminate, and then specialize the indeterminate to an arbitrary element $t \in K$. The result is a nonsingular matrix, again denoted $x(t;\ \alpha, R)$, or equivalently, a nonsingular linear transformation of $\mathfrak{M}$. Let G^R denote the linear group generated by the $x(t;\ \alpha, R)$ for all $t \in K$, $\pm\alpha \in \Pi$.

THEOREM 3. *If K is algebraically closed, then G^R is an irreducible algebraic group.*

Proof. We first observe that the mapping $t \rightarrow x(t;\ \alpha, R)$ for a fixed α is a homomorphism of $(K, +)$ into G^R. For any complex numbers t, t' we have $x(t + t';\ \alpha, R) = x(t;\ \alpha, R)\, x(t';\ \alpha, R)$. Replacing the complex parameters by indeterminates, we get a matrix identity in which the entries are polynomials with integer coefficients. Specialization to $t, t' \in K$ gives the fact that $t \to x(t;\, \alpha, R)$ is a homomorphism.

Let $X(\alpha, R)$ be the group of all $x(t;\, \alpha, R)$, $t \in K$. Then $X(\alpha, R)$ is the image of $(K, +)$ under the rational representation $t \to x(t; \alpha, R)$. Hence $X(\alpha, R)$ is an irreducible algebraic group [2, pp. 115, 112]. Since G^R is generated by irreducible algebraic groups, it is also irreducible algebraic [2, p. 123].

6. **The fixed point theorem.** For the time being, let $\mathfrak{L}$ denote a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic 0. Let $\mathfrak{M}$ be a finite-dimensional irreducible $\mathfrak{L}$-module with associated representation R, and let k denote the multiplicity of 0 as a weight of $\mathfrak{M}$. For each $a \in \mathfrak{L}$, let $\mathfrak{Z}_a = \{z \in \mathfrak{M} \mid za^R = 0\}$.

Lemma 2. *$\dim \mathfrak{Z}_a \geqq k$ for each $a \in \mathfrak{L}$.*

Proof. First let a be a regular element. Then the Fitting null component $\mathfrak{H}_a$ of $\operatorname{ad} a$ is a Cartan subalgebra containing a [8, p. 59]. Let $\mathfrak{M}_0$ denote the 0 weight space of $\mathfrak{M}$ relative to $\mathfrak{H}_a$. Then $\mathfrak{M}_0 \subseteq \mathfrak{Z}_a$, and $k = \dim \mathfrak{M}_0 \leqq \dim \mathfrak{Z}_a$.

Now let $a_1, a_2, \cdots, a_n$ be a basis for $\mathfrak{L}$, and let $\xi_1, \xi_2, \cdots, \xi_n$ be algebraically independent indeterminates. Let $\mathrm{P} = \Phi(\xi_1, \cdots, \xi_n)$, the field of rational expressions in the ξ_i, and consider $\mathfrak{L}_{\mathrm{P}}$ and $\mathfrak{M}_{\mathrm{P}}$ obtained by extension of the base field. The generic element $x = \sum \xi_i a_i$ of $\mathfrak{L}$ is a regular element of $\mathfrak{L}_{\mathrm{P}}$, hence also of $\mathfrak{L}_\Omega$, where Ω is the algebraic closure of P [8, pp. 66-61]. If $\mathfrak{Z}_x$ denotes the null space of x^R in $\mathfrak{M}_\Omega$ (where R is used to denote the extension of itself), then $\dim_\Omega \mathfrak{Z}_x \geqq k$, by the first part of the proof. Hence rank $x^R = \dim \mathfrak{M}_\Omega - \dim \mathfrak{Z}_x \leqq \dim \mathfrak{M}_\Omega - k = \dim_\Phi \mathfrak{M} - k$. An arbitrary element $a \in \mathfrak{L}$, say $a = \sum t_i a_i$, is the image of x under the specialization $\xi_i \to t_i$. Thus rank $a^R \leqq \operatorname{rank} x^R \leqq \dim \mathfrak{M} - k$, so $\dim \mathfrak{Z}_a \geqq k$.

Let e_α denote a root vector of $\mathfrak{L}$ for each (nonzero) root α relative to a Cartan subalgebra $\mathfrak{H}$. Let $\xi_1, \cdots, \xi_r$ be indeterminates, and let P now denote the field of formal power series over Φ in the ξ_i's (i.e., the quotient field of the algebra of formal power series). Consider $\mathfrak{L}_{\mathrm{P}}$ and the nonsingular linear transformation $\tau(\xi) = (\exp \xi_1 e_{\alpha_1}^R) \cdots (\exp \xi_r e_{\alpha_r}^R)$ of $\mathfrak{M}_{\mathrm{P}}$, where the α_i are arbitrary roots. Let $\mathfrak{N}$ be the fixed point space of $\tau(\xi)$ in $\mathfrak{M}_{\mathrm{P}}$.

Lemma 3. *$\dim \mathfrak{N} \geqq k$.*

Proof. Form the free algebra $\mathfrak{X}$ and the free Lie algebra $\mathfrak{F}$ on r generators $\eta_1, \cdots, \eta_r$. $\mathfrak{X}$ may be extended to the algebra $\overline{\mathfrak{X}} = \Phi[[\eta_1, \cdots, \eta_r]]$ of formal power series in the η_i's, and $\mathfrak{F}$ may be extended to the Lie subalgebra $\overline{\mathfrak{F}}$ of power series, each whose homogeneous terms is a Lie element of $\mathfrak{X}$, i.e., an element of $\mathfrak{F}$. Let $\tau(\eta) = (\exp \eta_1) \cdots (\exp \eta_r) \in \overline{\mathfrak{X}}$. The Campbell-Hausdorff formula [8, p. 172] gives $\tau(\eta) = \exp \zeta$, where $\zeta \in \overline{\mathfrak{F}}$. The canonical mapping $\overline{\mathfrak{F}} \to \Phi[[\xi_1 e^R_{\alpha_1}, \cdots, \xi_r e^R_{\alpha_r}]]$ maps $\tau(\eta) \to \tau(\xi)$ and $\exp \zeta \to \exp A$, where A is a Lie element, i.e., a power series in the elements of the Lie algebra generated by $\xi_1 e^R_{\alpha_1}, \cdots, \xi_r e^R_{\alpha_r}$. Since $\operatorname{ad} e_{\alpha_i}$ is nilpotent in $\mathfrak{L}$, ad $\xi_i e^R_{\alpha_i}$ is nilpotent in $\mathfrak{L}^R_{\mathrm{P}}$. Hence one may apply the Specht-Wever Theorem (see e.g. [8, pp. 169, 173]) to show that there are only a finite number of nonzero terms in the power series A. Hence $A \in \mathfrak{L}^K_{\mathrm{P}}$, which means $A = a^K$ for some $a \in \mathfrak{L}_{\mathrm{P}}$, and $\tau(\xi) = \exp a^K$. Now let $z \in \mathfrak{Z}_a = \{ z \in \mathfrak{M}_{\mathrm{P}} \mid z a^K = 0 \}$. Then $z\tau(\zeta) = z$, and hence $\mathfrak{Z}_a \subseteq \mathfrak{N}$. If Ω denotes the algebraic closure of P, we have, by Lemma 2, $\dim_{\mathrm{P}} \mathfrak{Z}_a = \dim_\Omega (\mathfrak{Z}_a)_\Omega \geqq k$, and therefore $\dim_{\mathrm{P}} N \geqq k$.

We now resume the notation of § 5. $\mathfrak{L}_C$ denotes a complex semi-simple Lie algebra, and $\mathfrak{L}$ the associated Lie algebra over an arbitrary field K of characteristic $\neq 2, 3$. Let $\mathfrak{M}_C$ be an irreducible $\mathfrak{L}_C$-module and $\mathfrak{M}$ the associated $\mathfrak{L}$-module. Let R be the corresponding representation in either case, and let G^K be the group defined in §5. For each $\tau \in G^K$, let $\mathfrak{F}(\tau)$ denote the fixed point space of τ in M.

Theorem 4. *For each $\tau \in G^K$, $\dim \mathfrak{F}(\tau) \geqq k$, the multiplicity of* 0 *as a weight of $\mathfrak{L}_C$ in $\mathfrak{M}_C$.*

Proof. Write $\tau = x(t_1; \alpha_1, R) \cdots x(t_r; \alpha_r, R)$. As in Lemma 3, let $\xi_1, \cdots, \xi_r$ be indeterminates, P the field of formal power series over C in the ξ_i and $\tau(\xi) = (\exp \xi_1 e^R_{\alpha_1}) \cdots (\exp \xi_r e^R_{\alpha_r})$. The element τ of G^R is obtained by observing that relative to a Chevalley basis for $\mathfrak{M}_C$ (which is also a basis for $(\mathfrak{M}_C)_{\mathrm{P}}$) the matrix $\tau(\xi)$ has entries which are polynomials in the ξ_i's with coefficients in the ring B defined in §5, and then specializing $\xi_i \to t_i$ in K. By Lemma 3, the space $\mathfrak{N}$ of $\tau(\xi)$-fixed points in $(\mathfrak{M}_C)_{\mathrm{P}}$ has dimension $\geqq k$. Thus rank $(\tau(\zeta) - I) \leqq \dim (\mathfrak{M}_C)_{\mathrm{P}} - k = \dim_K \mathfrak{M} - k$. Hence rank $(\tau - I) = \dim \mathfrak{M} - \dim \mathfrak{F}(\tau) \leqq \dim \mathfrak{M} - k$, and $\dim \mathfrak{F}(\tau) \geqq k$.

Corollary. *If* 0 *is a weight of R (in $\mathfrak{M}_C$), then every element of G^K has a (nonzero) fixed point.*

The following criterion is useful in determining when the fixed point theorem is applicable.

Theorem (Freudenthal [4]). *Zero is a weight of $\mathfrak{M}_C$ if and only if the highest weight is a sum of fundamental roots.*

If we take our usual basis $h_1, h_2, \cdots, h_l$ for $\mathfrak{H}_C$, each dominant integral linear function Λ on $\mathfrak{H}_C$ is a sum of the basic highest weights $\lambda_1, \lambda_2, \cdots, \lambda_l$ defined by $\lambda_i(h_j) = \delta_{ij}$. Thus one can determine whether 0 is a weight of $\mathfrak{M}_C$ for a given Λ by examining the coefficients of the λ_i written as linear combinations of the α_i (see e.g. [5, p. 318]).

7. **Automorphisms of exceptional simple Jordan algebras.** Application of Theorem 4 to the 26-dimensional irreducible module for the Lie algebra of type F_4 will be seen to give information about the fixed points of automorphisms of exceptional simple Jordan algebras. We assume familiarity with the notation, computational formulas, and constructions in Seligman [10, §§1-2]. Unexplained notations are defined there.

Let $\mathfrak{J}$ be the split exceptional simple Jordan algebra over an arbitrary field K (which will continue to mean characteristic $\neq 2, 3$). Let $\mathfrak{L}$ be the simple Lie algebra over K of classical type F_4. Then $\mathfrak{L}$ can be represented as the derivation algebra of $\mathfrak{J}$. We take as a fundamental system of roots for $\mathfrak{L}$ the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ defined on p. 293 of [10]. Let $\mathfrak{L}_\alpha$ denote the root space of α in $\mathfrak{L}$. Then the elements h_i' of $[\mathfrak{L}_{\alpha_i}, \mathfrak{L}_{-\alpha_i}]$ such that $\alpha_i(h_i') = 2$ $(i = 1,2,3,4)$ are given by:

$$h_1' = h_2 - h_3,$$
$$h_2' = h_1 - h_2,$$
$$h_3' = 2h_1,$$
$$h_4' = -h_1 + h_2 + h_3 + h_4,$$

as may be verified by [10, (14), (15), (16), (18)]. A set of root vectors e_i, f_i $(1 \leqq i \leqq 4)$ for the $\pm\alpha_i$ such that $\{e_i, f_i, h_i' \mid 1 \leqq i \leqq 4\}$ is a canonical set of generators is given by:

$$e_1 = 2(0,0,0,E_{32} - E_{67}), \qquad f_1 = 2(0,0,0,E_{76} - E_{23}),$$
$$e_2 = 2(0,0,0,E_{16} - E_{25}), \qquad f_2 = 2(0,0,0,E_{52} - E_{61}),$$
$$e_3 = 2(u_1,0,0,0), \qquad f_3 = (u_5,0,0,0),$$
$$e_4 = 2(0,u_5,0,0), \qquad f_4 = (0,u_1,0,0).$$

Again, the fact that $[e_if_i] = h_i'$ for each i can be verified by [10, (14)]. For calculating the effect of the e_i and f_i as derivations of $\mathfrak{J}$, one also needs to observe that $(E_{32} - E_{67})^\psi = (E_{32} - E_{67})^\phi = E_{32} - E_{67}$, $(E_{16} - E_{25})^\psi = (E_{16} - E_{25})^\varphi = 4(E_{83} - E_{74})$ and similar expressions for the effects of φ and ψ on the skew transformations appearing in the definitions of f_1 and f_2. This follows from [10, pp. 293-294].

We select the following fixed basis for $\mathfrak{J}$: 1, $w_1 = \operatorname{diag}\{1, -1, 0\}$, $w_2 = \operatorname{diag}\{0, 1, -1\}$, $2^{m(4)}u_1(1, 2), \cdots, 2^{m(11)}u_8(1, 2)$, $2^{m(12)}u_1(1,3), \cdots$, $2^{m(19)}u_8(1,3)$, $2^{m(20)}u_1(2,3), \cdots, 2^{m(27)}u_8(2,3)$, where the exponents $m(j)$ take the following values for $j = 4, 5, \cdots, 27$: $0, -1, -1, 4, -1, 0, 0, -5; -1, -2, -2, 3, 0, 1, 1, -4; 3, 2, 2, -1, -4, -3, -3, 0$.

Let (i,j) denote a 27-by-27 matrix unit with 1 in the (i,j)-position. Then relative to this basis for $\mathfrak{J}$, the e_i and f_i have the following matrices [10; (20)]:

$$e_1 = (6,5) - (9,10) + (14,13) - (17,18) + (22,21) - (25,26),$$

$$f_1 = (10,9) - (5,6) + (18,17) - (13,14) + (26,25) - (21,22),$$

$$e_2 = (4,9) - (5,8) + (19,14) - (18,15) + (27,22) - (26,23),$$

$$f_2 = (8,5) - (9,4) + (15,18) - (14,19) + (23,26) - (22,27),$$

$$e_3 = 2(2,4) - (3,4) - (8,2) - (13,26) + (14,25) - (15,20)$$
$$+ (16,27) - (21,18) + (22,17) - (23,12) + (24,19),$$

$$f_3 = 2(2,8) - (3,8) - (4,2) + (12,23) - (17,22) + (18,21)$$
$$- (19,24) + (20,15) - (25,14) + (26,13) - (27,16),$$

$$e_4 = (2,16) + (3,16) + (4,27) + (9,22) - (10,21) + (11,24)$$
$$- (12,2) - (12,3) - (20,7) - (23,8) + (25,6) - (26,5),$$

$$f_4 = (2,12) + (3,12) + (5,26) - (6,25) + (7,20) + (8,23)$$
$$- (16,2) - (16,3) + (21,10) - (22,9) - (24,11) - (27,4).$$

From these expressions we may compute directly:

$$e_1^2 = f_1^2 = e_2^2 = f_2^2 = 0,$$

$$e_3^2 = -2(8,4), f_3^2 = -2(4,8),$$

$$e_4^2 = -2(12,16), f_4^2 = -2(16,12),$$

$$e_3^3 = f_3^3 = e_4^3 = f_4^3 = 0.$$

THEOREM 5. *Every automorphism of an exceptional central simple Jordan algebra $\mathfrak{J}$ over an arbitrary field of characteristic $\neq 2, 3$ has at least a 3-dimensional space of fixed points.*

Proof. Let Ω be the algebraic closure of the base field K. Then $\mathfrak{J}_\Omega$ is a necessarily split exceptional simple Jordan algebra over Ω, and since fixed point spaces are preserved under field extension, it suffices to prove the theorem in the algebraically closed case. Thus we may assume $\mathfrak{J}$ itself is split.

Let $\mathfrak{J}_C$ denote the exceptional simple Jordan algebra over the complex field, and $\mathfrak{L}_C$ its derivation algebra. The restriction of $\mathfrak{L}_C$ to the space $\mathfrak{J}'_C$ of elements of trace 0 gives a 26-dimensional irreducible representation of $\mathfrak{L}_C$, so we set $\mathfrak{M}_C = \mathfrak{J}'_C$. If we apply the previous contents of this section to $\mathfrak{J}_C$, we see that the basis for $\mathfrak{M}_C$ obtained by dropping the first basis element for $\mathfrak{J}_C$ is in fact a Chevalley basis relative to the selected canonical generators for $\mathfrak{L}_C$. The full basis for $\mathfrak{J}_C$ has a multiplication table with coefficients in the ring B. Thus when we pass to our arbitrary field K, we get not only $\mathfrak{L}$ and $\mathfrak{M}$, but also the split exceptional Jordan algebra $\mathfrak{J}$. Furthermore,

$\mathfrak{M} = \mathfrak{J}'$, the space of elements of trace 0 in $\mathfrak{J}$, and $\mathfrak{L}$ is the Lie algebra of derivations of $\mathfrak{J}$.

The zero weight space in $\mathfrak{M}_C$ is spanned by w_1 and w_2. Thus Theorem 4 asserts that every element of the group G^R generated by the mappings $x(t;\alpha,R)$ $(t \in K, \pm\alpha \in \Pi)$ of $\mathfrak{M}$ has at least a 2-dimensional fixed point space.

We note that e_α^R denotes a nilpotent linear transformation of $\mathfrak{M}_C$ (or $\mathfrak{M}$); it may be extended uniquely to a derivation of $\mathfrak{J}_C$ (or $\mathfrak{J}$) by setting $1e_\alpha^R = 0$. Similarly, $x(t;\alpha,R)$ $(t \in C)$ will also denote the extension of itself to $\mathfrak{J}_C$ defined by $1 \cdot x(t;\alpha,R) = 1$. Clearly, this is the exponential of the extended te_α^R, a nilpotent derivation, and hence is an automorphism of $\mathfrak{J}_C$. Passing from C to K in the usual way gives automorphisms $x(t;\alpha,R)$ of $\mathfrak{J}$. In this way G^R can be considered as a subgroup of $\mathfrak{A}(\mathfrak{J})$, the automorphism group of $\mathfrak{J}$. since each element of G^R has at least a 3-dimensional fixed point space in $\mathfrak{J}$, the proof will be complete when we show that $G^R = \mathfrak{A}(\mathfrak{J})$.

If $A \in \mathfrak{A}(\mathfrak{J})$, then I_A: $X \to A^{-1}XA$ for $X \in \mathfrak{L}$ defines an automorphism of $\mathfrak{L}$, and $A \to I_A$ is a monomorphism of $\mathfrak{A}(\mathfrak{J})$ into $\mathfrak{A}(\mathfrak{L})$. The image of $x(t;\alpha,R)$ is $x(t;\alpha)$, $\pm\alpha \in \Pi$, $t \in K$ (by [9, p. 454, (2)], since in this case $(e_\alpha^R)^3 = 0$, and $x(t;\alpha,R) = \exp te_\alpha^R$). The Chevalley group G' is generated by the $x(t;\alpha)$ for $\pm\alpha \in \Pi$, $t \in K$, and in this case is the full automorphism group of $\mathfrak{L}$ [12, (4.8)], so $A \to I_A$ maps G^R onto $\mathfrak{A}(\mathfrak{L})$ and therefore

$$G^R = \mathfrak{A}(\mathfrak{J}) \cong \mathfrak{A}(\mathfrak{L}).$$

The next task is to show that the lower bound in Theorem 5 is actually achieved. For this purpose, familiarity with the notation and terminology of Jacobson [7] (particularly the proof of the triality principle in §2) is assumed. Theorem 6 will be proved by exhibiting an automorphism that achieves the lower bound, in a manner analogous to the proof of Theorem 2 of [11].

THEOREM 6. *The minimal dimension of fixed point spaces of automorphisms of a split exceptional simple Jordan algebra $\mathfrak{J}$ over a field K of characteristic $\neq 2, 3$, or 5 is 3.*

Proof. Continuing the notation of [10] with respect to $\mathfrak{J}$ and the split Cayley algebra $\mathfrak{C}$ over K, we select the following elements $b_1, \cdots, b_8$ of $\mathfrak{C}$:

$$b_1 = u_4 + u_8, \qquad b_2 = u_4 - \frac{1}{2}u_8;$$

$$b_{2i+1} = 2u_i - \frac{1}{2}u_{i+4}, \qquad i = 1, 2, 3;$$

$$b_{2i+2} = -u_i - \frac{1}{2}u_{i+4}, \qquad i = 1, 2, 3.$$

The multiplication table for $\mathfrak{C}$ [10, p. 287] may be used to calculate the matrices of the symmetries S_1, S_{b_i}, with respect to the identity and to the b_i, and we find that

$$A_1 \equiv \prod_{i=1}^{8} S_1 S_{b_i} = \operatorname{diag}\left\{-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -2, -2, -2, -2\right\}.$$

The orthogonal transformations A_2, A_3 related to A_1 by the triality principle [7, p. 78, (13)] are given by

$$A_2 = \beta \prod_{i=1}^{8} n(b_i)^{-1} b_{iL}, \qquad A_3 = \beta^{-1} \prod_{i=1}^{8} b_{iR},$$

where $\beta^2 = \prod_{i=1}^{8} n(b_i)$ [7, pp. 75, 79]. We have $n(b_i) = 2$ for i odd and -1 for i even, so $\prod_{i=1}^{8} n(b_i) = 16$. We take $\beta = -4$. The left and right multiplications by the b_i may also be computed from the multiplication table, and we find:

$$A_2 = \operatorname{diag}\left\{2, 2, 2, 2, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\},$$

$$A_3 = \operatorname{diag}\left\{-1, -1, -1, -\frac{1}{4}, -1, -1, -1, -4\right\}.$$

A_1, A_2, and A_3 define an automorphism of $\mathfrak{J}$ [7, p. 87, Proof of Theorem 6] whose fixed point space is the 3-dimensional space spanned by the diagonal idempotents. This, together with Theorem 5, completes the proof.

Other simple cases in which the fixed point theorem can be applied are the irreducible representations of minimal degree for $\mathfrak{L}_C$ of types B_l and G_2. In the first case one gets the known result that every rotation in an odd-dimensional space over an arbitrary field (here of characteristic $\neq 2, 3$) has a nonzero fixed point [1, p. 131]. In the second case, the result is that every automorphism of a Cayley algebra has a fixed point in the 7-dimensional subspace of elements of trace 0. This result is also known, and in fact is a special case of the previous one, since automorphisms of Cayley algebras act in the trace 0 space as rotations with respect to the norm form [6]. In each case, the details of applying the fixed point theorem are similar to the F_4 case worked out above. Application of Theorem 4 to adjoint representations also yields a part of the result of [11, Theorem 1].

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